Fuzzy Propositional Logic Associated with Quantum Computational Gates

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We apply residuated structures associated with fuzzy logic to develop certain aspects of information processing in quantum computing from a logical perspective. For this purpose, we introduce an axiomatic system whose natural interpretation is the irreversible quantum Poincaré structure.

1. INTRODUCTION

This paper is framed in a wide program of the application of fuzzy logic to analyze quantum information. In Domenech and Freytes (2005), we presented a logical formalization associated with contextuality using Heyting structures of the intuitionistic calculus. They are a particular case of residuated lattices (Kowalski and Ono, 2000), which are the algebraic counterpart of some systems of fuzzy logic. In the same line, we apply here residuated structures to develop certain aspects of information processing in quantum computing from a logical perspective.

Recently, increasing attention has been paid to quantum computational logic (QCL) (Cattaneo *et al.*, 2004; Dalla Chiara *et al.*, 2003; Gudder, 2002), a logic related with quantum computation (Lomonaco, 2002; Nielsen and Chuang, 2000), different from quantum logic (QL). On one hand, QCL differs from QL first introduced by Birkhoff and von Neumann (1936) in that it is not an orthomodular structure. On the other hand, they have different semantics. A sentence in QL may

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be interpreted as a closed subspace of the Hilbert space of the system, the space of its possible pure states. Instead, the meaning of an elementary sentence in QCL is a quantum information quantity encoded in a collection of qbits (unit vectors pertaining to the tensorial product of two-dimensional complex Hilbert spaces that represent pure states) or in qmixes (positive semi-definite Hermitian operators of trace one on that space, which represent statistical mixtures of pure states). Moreover, conjunction and disjunction have to be interpreted otherwise than in QL because they are not associated to join and meet lattice operations. Instead, the number of conjunctions and disjunctions involved in a sentence determines the dimension of the space of its "meanings": this space is not the fixed dimensional Hilbert space of the states of the system as in QL but a tensorial product of two-dimensional Hilbert spaces in which the number of factors varies according to the nature and number of logical connectives. So, the "meaning" of a sentence reflects the logical form of the sentence itself, and this has been pointed as an intensional character of QCL (Dalla Chiara *et al.*, 2003). It also has new connectives that show a genuine quantum behavior, not only when applied to superpositions but also transforming each element of the logical basis in superpositions.

Our work is motivated in two questions posed in Cattaneo *et al.* (2004) and Dalla Chiara *et al.* (2003). One of them mentions that QCL may be seen as a "new example of fuzzy logics." In this direction, we are interested in the logical formalization of the treatment of quantum information during the computational process. More precisely, we want to establish a lowest bound that allows to appreciate the relevance of inputs that are known with certainty with respect to the possible outputs. The frame we use for this development is the *approximate reasoning*, which is a crucial theme studied within fuzzy logic. In order to accomplish these purposes we must introduce an axiomatization for QCL, which is the second question, referred to in Cattaneo *et al.* (2004), as an open problem. This system allows to establish a completeness theorem for the mentioned treatment of information. We first show the relation between QCL and fuzzy logic in a rigorous manner, more precisely its relation with the infinite valued enriched Lukaciewicz calculus. This motivates a way to axiomatization of QCL that reflects the minimum set of basic properties associated to the quantum gates. This axiomatization allows in turn to establish the completeness result mentioned above.

The paper is organized as follows. Section 2 summarizes basic notions about quantum gates involved in QC. In Section 3, we present nonunitary operations that may be performed on states of physical systems and the Poincaré structure which will be the natural model used as interpretation of the logical system associated with QC. In Section 4, we present some preliminary on universal algebra and MV- and PMV-algebras. These structures are the algebraic counterparts of infinite many valued logics. It is necessary here to study injective objects in the variety of PMV-algebras with the purpose to obtain tools for a completeness theorem based on approximate reasoning to be developed later. In Section 5, we outline

the relations between the logic associated to the Poincaré structure and fuzzy logic. In Section 6, we introduce an axiomatic system for the irreversible quantum computational structure and show basic results on it. Section 7 is dedicated to the study of a fragment of the mentioned logic. From this fragment, in Section 8, we show the formal connection with fuzzy logic and establish a Pavelka style completeness theorem (Hájek, 1998; Pavelka, 1979) and a type of compactness theorem. Finally, Section 9 is devoted to the conclusions.

2. BASIC NOTIONS IN THE THEORY OF QUANTUM COMPUTATION

2.1. Qbits and Qmixes

In quantum computation, information is elaborated and processed by means of quantum systems. Let us first consider the two-dimensional case in which pure states of a system are coherent superpositions of two distinguishable states (may be polarization states of a single photon, spin states of an electron, and so on) represented by unit vectors in the two-dimensional complex Hilbert space $\mathcal{C}^2$, and called *qbits*. The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{C}^2$ is called the *logical basis*. This name refers to the fact that truth is related with $|1\rangle$ and falsity with $|0\rangle$. A qbit $|\varphi\rangle$ may be written as a linear superposition of the basis vectors with complex coefficients (*probability amplitudes*):

$$|\varphi\rangle = c_0|0\rangle + c_1|1\rangle, \quad \text{with } |c_0|^2 + |c_1|^2 = 1$$

Associated with each basis vector is a projector onto the subspace it spans, i.e. $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$. Because of the usual noncomplete efficiency of the preparation procedure of physical systems, quantum states are better described by statistical mixtures. They are represented by density operators ρ acting on $\mathcal{C}^2$, called *qmixes*. Qmixes are decomposable in terms of the mentioned projector operators:

$$\rho_\lambda = (1 - \lambda)P_0 + \lambda P_1, \quad \lambda \text{ a real number } \in [0, 1]$$

and include pure states as particular cases. We call $\mathcal{D}(\mathcal{C}^2)$ the set of density operators on $\mathcal{C}^2$.

Quantum mechanics reads out the information content of a pure state via the Born rule. By these means, a *probability value* which we will later be interested in may be assigned to a qbit or a qmix in the following way (Dalla Chiara *et al.*, 2003):

Definition 2.1. Let $|\varphi\rangle = c_0|0\rangle + c_1|1\rangle$ be a qbit, i.e. a unit vector in $\mathcal{C}^2$. Then its *probability value* $p(|\varphi\rangle)$ is $p(|\varphi\rangle) = |c_1|^2$, the square module of the amplitude corresponding to the basis vector associated with truth.

Definition 2.2. Let $\rho_\lambda = (1 - \lambda)P_0 + \lambda P_1 \in \mathcal{D}(\mathcal{C}^2)$ be a qmix. Then its *probability value* $p(\rho_\lambda)$ is $p(\rho_\lambda) = \lambda$, the coefficient corresponding to the projector associated with truth.

In order to take into account computations in which multiple qbits are involved, we have to extend the two-dimensional Hilbert space to the state space of compound systems. For n qbits, this is performed by taking the n-fold tensor product of state spaces of one qbit. Thus, a unit vector in $\mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \cdots \otimes \mathcal{C}^2 \equiv \otimes^n \mathcal{C}^2$ represents the pure state of an *n-qbit* system. We use x_i, $1 \le i \le n$, as variables ranging over the set $\{0, 1\}$ and vectors pertaining to each of the n-logical basis are denoted by $|x_i\rangle$. So, $|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle (|x_1, \ldots, x_n\rangle$ for short) is a factorized vector of $\otimes^n \mathcal{C}^2$ and $\mathcal{B}^{(n)} = \{|x_1, \ldots, x_n\rangle, x_i \in \{0, 1\}\}$ is an orthonormal basis for the n-qbits system, called the *computational basis*. Recall that the dimension of $\otimes^n \mathcal{C}^2$ is 2^n. Then, pure states of n-qbits systems are normalized linear combinations of the elements of this basis.

To represent qmixes in product spaces, we use the same notation as (Dalla Chiara *et al.*, 2003). So, let us consider the following sets of natural numbers:

$$C_1^{(n)} := \{i : | |i\rangle\rangle = |x_1, x_2, \ldots, x_n\rangle, \quad \text{with } x_n = 1\}$$

$$C_0^{(n)} := \{i : | |i\rangle\rangle = |x_1, x_2, \ldots, x_n\rangle, \quad \text{with } x_n = 0\}$$

Any n-qbit $|\mu\rangle \in \otimes^n \mathcal{C}^2$

$$|\varphi\rangle = \sum_{i=0}^{2^n - 1} a_i | |i\rangle\rangle$$

may be expressed as

$$|\varphi\rangle = \sum_{i \in C_0^{(n)}} a_i | |i\rangle\rangle + \sum_{j \in C_1^{(n)}} | |j\rangle\rangle$$

$P_0^{(n)}$ and $P_1^{(n)}$ are the projectors onto the subspaces spanned by $\{| |i\rangle\rangle : i \in C_0^{(n)}\}$ and $\{| |i\rangle\rangle : i \in C_1^{(n)}\}$, respectively. The corresponding density operators associated to them are $k_n P_0^{(n)}$ and $k_n P_1^{(n)}$, with

$$k_n = \frac{1}{2^n - 1}.$$

Let now $\mathcal{D}(\otimes^n \mathcal{C}^2)$ be the set of all density operators of $\otimes^n \mathcal{C}^2$ and

$$\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}(\otimes^n \mathcal{C}^2)$$

Then,

Definition 2.3.   A qmix is a density operator in $\mathcal{D}$.

The probability value of an *n*-qmix is also given by the Born rule:

Definition 2.4.   For any $\rho \in \mathcal{D}$, the *probability value* of ρ is

$$p(\rho) = \mathrm{Tr}\big(P_1^{(n)}\rho\big)$$

The quantum computation process may be summarized as follows: first an initial state of a physical system (a qmix) is provided as the *input*. Then, it evolves according to the *elementary operations* that are performed on it, each one associated to a quantum gate. Finally, the access to the information content of the resulting state is possible via the measurement operation that provides one of the possible results, more about this below when we deal with approximate reasoning.

2.2. Quantum Gates on Qbits

In QCL, logical connectives are interpreted as transition operations on the physical system and represented by quantum gates. Quantum gates are implemented as processes that transform the state of the system at a certain initial time t_0 to its state at a later final time t. Because of the Schrödinger equation, this change is given by an unitary operator $U(t_0, t)$ if the evolution is not explicitly depending on time. So,

$$\rho_\lambda(t) = U(t_0, t)\rho_\lambda(t_0)U^\dagger(t_0, t)$$

where $U^\dagger$ is the adjoint of U. Unitary operations preserve vector lengths and angles. Moreover, they are reversible operations, so the process taking from $t_0 \longrightarrow t$ may be reversed without any loose of information, as Charles Bennett first announced in the early 1970s (Bennett, 1973), exception made of the reading of the result or the ignoring of some degrees of freedom, to what we will turn on later. We need to introduce here the following quantum gates acting on qbits:

Definition 2.5.   The **noop** gate applied on a qbit is the operation that does nothing. The **noop** gate is the identity operation. We will call it **I**. It is hence generalized to the case of *n*-qbits.

Definition 2.6.   The **not** (*negation*) gate applied on a qbit is the linear operation **not**: $\mathcal{C}^2 \rightarrow \mathcal{C}^2$ that acts over the logical basis in the following way:

$$\mathbf{not}|0\rangle = |1\rangle$$

$$\mathbf{not}|1\rangle = |0\rangle$$

Its corresponding matrix representation in the logical basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e. the σ_x Pauli matrix (see Section 3).

Definition 2.7. The **not**$^{(j)}$ (*negation of the j-register*) gate applied on an n-qbit is the linear operation **not**$^{(j)} : \otimes^n \mathcal{C}^2 \to \otimes^n \mathcal{C}^2$ that acts over any element of the computational basis in the following way:

$$\mathbf{not}^{(j)} |x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{j-1}, 1 - x_j, x_{j+1}, \ldots, x_n\rangle$$

Its corresponding $2^n \times 2^n$ matrix representation in the computational basis is the block matrix with identities in the diagonal everywhere except for the block acting on x_j, where **I** is replaced by the σ_x matrix Pauli.

Definition 2.8. The $\sqrt{\mathbf{not}}$ (*square root of the negation*) gate applied on a qbit is the linear operator $\sqrt{\mathrm{not}} : \mathcal{C}^2 \to \mathcal{C}^2$ that acts over the logical basis in the following way:

$$\sqrt{\mathbf{not}} \, |0\rangle = \frac{1 + \iota}{2} |0\rangle + \frac{1 - \iota}{2} |1\rangle$$

$$\sqrt{\mathbf{not}} \, |1\rangle = \frac{1 - \iota}{2} |0\rangle + \frac{1 + \iota}{2} |1\rangle$$

The corresponding matrix representation in the logical basis is

$$\begin{pmatrix} \dfrac{1 + \iota}{2} & \dfrac{1 - \iota}{2} \\ \dfrac{1 - \iota}{2} & \dfrac{1 + \iota}{2} \end{pmatrix}$$

Physically, it is a device that, whenever the input is, for example, $|0\rangle$ the output is

$$\begin{cases} |0\rangle, & \text{with probability } 1/2 \\ |1\rangle, & \text{also with probability } 1/2 \end{cases}$$

and analogously for $|1\rangle$. So it looks like a random switch. It may be thought that, applying it twice, it would go on behaving the same way. But this is not the case. Instead, applying it twice gives the usual negation **not**. This is precisely the property that characterizes $\sqrt{\mathbf{not}}$:

$$\sqrt{\mathbf{not}}\sqrt{\mathbf{not}}|\varphi\rangle = \mathbf{not}|\varphi\rangle$$

This is due to the fact that, when evaluating the probability of a quantum transition, we have to add the probability amplitudes of all possible paths and not the probabilities themselves. Probability amplitudes are complex numbers that may

cancel (destructive interference) or enhance each other (constructive interference) when summed up and this makes a big difference with classical probabilistic processes (see, for example, Aharonov, 1998; Deutsch *et al.*, 2000). The potential of quantum computing lies precisely in quantum interference among many computational paths. $\sqrt{\textbf{not}}$ is sometimes interpreted as a "tentative" negation: trying twice effectively produces negation (Dalla Chiara *et al.*, 2003).

   When applied to an *n*-qbit we have the following:

Definition 2.9. The $\sqrt{\text{not}}^{(j)}$ gate applied on an *n*-qbit is the linear operator $\sqrt{\text{not}}^{(j)} : \otimes^n \mathcal{C}^2 \to \otimes^n \mathcal{C}^2$ that acts on any $|\varphi\rangle$ in the following way:

$$\sqrt{\text{not}}^{(j)}|\varphi\rangle = \sum_{i=0}^{2^n-1} a_i |x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\rangle \otimes \sqrt{\text{not}}|x_j\rangle$$

Its matrix representation is constructed analogously to the **not** gate.

   To consider the (reversible) *conjunction*, the Toffoli gate is needed. Let us suppose that two multiple qbits $|\varphi\rangle \in \otimes^n \mathcal{C}^2$ and $|\phi\rangle \in \otimes^m \mathcal{C}^2$ are given. Denoting by $\otimes^{n+m+1} \mathcal{C}^2 \equiv (\otimes^n \mathcal{C}^2) \otimes (\otimes^m \mathcal{C}^2 \otimes \mathcal{C}^2)$,

Definition 2.10. The *Toffoli* gate $T^{(n,m,1)}$ is the linear operator

$$T^{(n,m,1)} : \otimes^{n+m+1} \mathcal{C}^2 \to \otimes^{n+m+1} \mathcal{C}^2$$

acting over any vector $|x_1, x_2, \ldots, x_n\rangle \otimes |y_1, y_2, \ldots, y_m\rangle \otimes |z\rangle$ in the computational basis in the following way:

$$T^{(n,m,1)}|x_1, x_2, \ldots, x_n\rangle \otimes |y_1, y_2, \ldots, y_m\rangle \otimes |z\rangle =$$

$$|x_1, x_2, \ldots, x_n\rangle \otimes |y_1, y_2, \ldots, y_m\rangle \otimes |\min(x_n, y_m) \oplus z\rangle$$

where $\oplus$ indicates the sum modulo 2.
On this basis, the (reversible) *conjunction* **and** may be defined.

Definition 2.11. For any $|\varphi\rangle \in \otimes^n \mathcal{C}^2$ and $|\phi\rangle \in \otimes^m \mathcal{C}^2$,

$$\textbf{and}\,(|\varphi\rangle, |\phi\rangle) := T^{(n,m,1)}|\varphi\rangle \otimes |\phi\rangle \otimes |0\rangle$$

We will deal little with **and** in what follows. We only remark here that the explicit writing in the computational basis of the *n*-qbits representing the pieces of information related by **and** reproduces the results in the truth table of the conjunction, added each file with the probability amplitude whose square modulus represents the corresponding probability of obtaining that result if measured. For considerations about this connective we refer to Cattaneo *et al.* (2004).

2.3. Quantum Gates on Qmixes

These gates are easily generalized to qmixes. As in Dalla Chiara *et al.* (2003), we use capital letters for gates acting on qmixes.

Definition 2.12.   The **NOOP** gate applied on an *n*-qmix $\rho \in \mathcal{D}$ does nothing.

Definition 2.13.   The **NOT** (*negation*) gate applied on a qmix $\rho_\lambda \in \mathcal{D}(\mathcal{C}^2)$ gives

$$\mathbf{NOT}\rho_\lambda = \mathbf{not}\rho_\lambda\mathbf{not}$$

As expected, **NOT** gate satisfies that.

Proposition 2.14.   *Let λ be a real number in $[0, 1]$. If we consider the density operator $\rho_\lambda = (1 - \lambda)P_0 + \lambda P_1 \in \mathcal{D}(\mathcal{C}^2)$ then we have*

$$p(\mathbf{NOT}\rho_\lambda) = 1 - \lambda$$

Definition 2.15.   The $\mathbf{NOT}^{(n)}$ gate applied on a density operator $\rho \in \mathcal{D}$ gives

$$\mathbf{NOT}^{(n)}\rho = \mathbf{not}^{(n)}\rho\mathbf{not}^{(n)}$$

Definition 2.16.   The $\sqrt{\mathbf{NOT}}$ (*square root of the negation*) gate applied on a qmix $\rho_\lambda \in \mathcal{D}(\mathcal{C}^2)$ gives

$$\sqrt{\mathbf{NOT}}\rho_\lambda = \sqrt{\mathbf{not}}\,\rho_\lambda\sqrt{\mathbf{not}}^\dagger$$

where the symbol † means the adjoint operation.
The previously discussed behavior of square root of the negation on qbits reflects for qmixes in the fact that

Proposition 2.17.   *Let λ be a real number in $[0, 1]$. If we consider the density operator $\rho_\lambda = (1 - \lambda)P_0 + \lambda P_1 \in \mathcal{D}(\mathcal{C}^2)$ then we have*

$$p(\sqrt{\mathbf{NOT}}\rho_\lambda) = \frac{1}{2}$$

Proof: See Dalla Chiara *et al.* (2003).   □

Definition 2.18.   The $\sqrt{\mathbf{NOT}}^{(n)}$ gate applied on a density operator $\rho \in \mathcal{D}$ gives

$$\sqrt{\mathbf{NOT}}^{(n)}\rho = \sqrt{\mathbf{not}}^{(n)}\rho\sqrt{\mathbf{not}}^{(n)\dagger}$$

The (reversible) *conjunction* **AND** of qmixes $\in \mathcal{D}(\mathcal{C}^2)$ is defined as follows:

Definition 2.19.   Consider $\tau_\lambda \in \mathcal{D}(\mathcal{C}^2)$ and $\nu_\gamma \in \mathcal{D}(\mathcal{C}^2)$, then

$$\mathbf{AND}(\tau_\lambda, \nu_\gamma) = T^{(1,1,1)}(\tau_\lambda \otimes \nu_\gamma \otimes P_0^{(1)}) T^{(1,1,1)}$$

and in the case of greater dimensions,

Definition 2.20.   Consider $\tau \in \mathcal{D}(\otimes^n \mathcal{C}^2)$ and $\nu \in \mathcal{D}(\otimes^m \mathcal{C}^2)$, then

$$\mathbf{AND}(\tau, \nu) = T^{(n,m,1)}(\tau \otimes \nu \otimes P_0^{(1)}) T^{(n,m,1)}$$

In order to access information after the computing process we apply the *measurement device* **meas**. It is not a unitary operation but an intrinsically probabilistic process that, when applied to a qbit, converts it into a probabilistic classical bit that is 0 (falsity) or 1 (truth) associated with the elements of the logical basis $\{|0\rangle, |1\rangle\}$ with probability equal to the modulus squared of its amplitude, i.e.

Definition 2.21.   Given $|\varphi\rangle = c_0|0\rangle + c_1|1\rangle$, **meas** device over it produces

$$\begin{cases} 0, |0\rangle, & \text{with probability } |c_0|^2 \\ 1, |1\rangle, & \text{with probability } |c_1|^2 \end{cases}$$

In the cases of qmixes $\rho_\lambda \in \mathcal{D}(\mathcal{C}^2)$, the results are:

$$\begin{cases} 0, P_0, & \text{with probability } (1 - \lambda) \\ 1, P_1, & \text{with probability } \lambda \end{cases}$$

In spite of the fact that quantum gates are unitary operations, quantum algorithms are not deterministic because of the randomness of the measurement process. Instead, there is a probability distribution of the possible outcomes. Nevertheless, they are useful because the output may be the correct solution to a very hard problem but such that the solution could be easily verified, as is the case of finding the prime factors of a large number—a hard problem whose solution is easily verifiable.

In what follows, we will deal with states represented by qmixes because they are the general case and include pure states.

3. POINCARÉ IRREVERSIBLE QUANTUM STRUCTURE

First we recall some properties of qbits and qmixes that we will use in what follows. We restrict first to $\mathcal{C}^2$. Let us consider the Bloch sphere, i.e. the sphere S^2 with unit radius:

$$S^2 = \left\{ x_i \in \mathcal{R}^3 \bigg/ \sum_{i=1}^{3} x_i^2 = 1 \right\}$$

The coefficients c_0 and c_1 of the decomposition in the logical basis of any qbit $|\varphi\rangle$ may be written as

$$c_0 = \cos \frac{\theta}{2}, \qquad c_1 = \exp(\iota\phi)\sin\frac{\theta}{2}$$

i.e. in terms of the Euler angles that determine a point of the Bloch sphere. Thus, a qbit may be seen as a point on the surface of the Bloch sphere. The transformation of the qbit due to the action of a gate in the computational process is a unitary transformation in $\mathcal{C}^2$, i.e. an element of $SU(2)$. So, because of the relation $SO(3) \cong SU(2)/Z_2$, a rotation in $\mathcal{R}^3$ corresponds to it. Thus, any transformation applied to the qbit may be considered a rotation.

Qmixes and any operator over $\mathcal{C}^2$ have a matrix representation

$$A = aI + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. In the logical basis,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In particular, because of the properties of density operators $\operatorname{Tr}\rho = 1$, $\rho^\dagger = \rho$, the matrix representation of ρ may be written as

$$\rho = \frac{1}{2}(I + r_1\sigma_x + r_2\sigma_y + r_3\sigma_z)$$

where r_1, r_2, r_3 are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$. Thus, each density operator ρ in $\mathcal{D}(\mathbf{C}^2)$ such that $\rho^2 = \rho$ (pure states) can be identified to a point (r_1, r_2, r_3) on the sphere and each ρ in $\mathcal{D}(\mathcal{C}^2)$ such that $\rho^2 < \rho$ (proper mixtures) with a point in the interior of the Bloch sphere. We denote this identifications as $\rho = (r_1, r_2, r_3)$.

Remark 3.1. We recall that the probability value of a qmix ρ in $\mathcal{D}(\mathcal{C}^2)$ is a function $p : \mathcal{D}(\mathcal{C}^2) \to [0, 1]$ determined by the Born rule $p(\rho) = \operatorname{Tr}(\rho P_1)$. When expanding ρ in terms of the Pauli matrices, it results that it depends only on the r_3 coefficient. So, the semantic consequence will be only associated with r_3. Moreover, it may be easily seen that the probability value of $p(\sqrt{\mathbf{NOT}}\rho)$ is only related to the r_2 coefficient of the expansion.
Precisely,

Lemma 3.2. [*Dalla Chiara et al., 2003, Lemma* 6.1]) *Let* $\rho = (r_1, r_2, r_3) \in \mathcal{D}(\mathcal{C}^2)$. *Then we have*

   1. $p(\rho) = \frac{1-r_3}{2}$,
   2. $p(\sqrt{\mathbf{NOT}}\rho) = \frac{1-r_2}{2}$.

Up to now we have been dealing with reversible gates representing unitary evolutions. But there are other interesting operations that may be done with qbits or qmixes. Namely, ignoring some degrees of freedom of the system and paying attention only to the others, what is performed by taking partial traces over the density operator of the system. This is, the same as **meas**, an irreversible operation and, in the case of taking a partial trace over pure states, this operation transforms it in a mixture.

Let us consider the state space $(\otimes^n \mathcal{C}^2) \otimes (\otimes^m \mathcal{C}^2) \otimes (\otimes^r \mathcal{C}^2)$. For any state ρ, there is a unique density operator $\tilde{\rho} = \mathrm{Tr}_{1,2}(\rho)$, the partial trace of ρ, called its *reduce state*, that represents the state obtained by ignoring the information related to the degrees of freedom in $\otimes^n \mathcal{C}^2$ and in $\otimes^m \mathcal{C}^2$. The states ρ and $\tilde{\rho}$ are statistically equivalent with respect to the subsystem whose state space is $\otimes^r \mathcal{C}^2$, in the sense that the mean value $\langle O \rangle$ of an observable O associated with this subsystem may be calculated by using any of them. That is, let $O : \otimes^r \mathcal{C}^2 \to \otimes^r \mathcal{C}^2$ and let $\tilde{O} = I^{(n)} \otimes I^{(m)} \otimes O : (\otimes^n \mathcal{C}^2) \otimes (\otimes^m \mathcal{C}^2) \otimes (\otimes^r \mathcal{C}^2) \to (\otimes^n \mathcal{C}^2) \otimes (\otimes^m \mathcal{C}^2) \otimes (\otimes^r \mathcal{C}^2)$ be its extension to the complete space state, then $\langle O \rangle = \mathrm{Tr}(\tilde{\rho} O) = \mathrm{Tr}(\rho \tilde{O})$. This consideration of the reduced state allows the definition of another conjunction, besides the reversible conjunction represented by the Toffoli gate, called *irreversible conjunction* **IAND**.

Definition 3.3. Let τ and ν in $\mathcal{D}(\mathcal{C}^2)$. Then the *irreversible conjunction* is defined as

$$\mathbf{IAND}\,(\tau, \nu) = \rho_{p(\tau)p(\nu)}$$

It may be seen that this operation is equivalent to a partial trace over the state **AND**(τ, ν) representing the reversible conjunction of the states τ, ν of two systems (Dalla Chiara *et al.*, 2003):

Proposition 3.4. *Consider the reversible conjunction*

$$\mathbf{AND}\,(\tau, \nu) = T^{(1,1,1)}(\tau, \nu, P_0)T^{(1,1,1)} \in \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$$

then **IAND**$(\tau, \nu) = \mathrm{Tr}_{1,2}(\mathbf{AND}(\tau, \nu))$

Another connective, namely the *Lukasiewicz disjunction*, may be defined, which is associated with an irreversible operation.

Definition 3.5. Let τ and ν in $\mathcal{D}(\mathcal{C}^2)$. Then, the *Lukasiewicz disjunction* is defined as

$$\tau \oplus \nu = \rho_{p(\tau) \oplus p(\nu)}$$

From now on and for simplicity in the logic algebraic handling, we will use the following conventions: $\neg\rho$ is the abbreviation for $(\mathbf{NOT}\rho)$, $\sqrt{\rho}$ is the abbreviation for $(\sqrt{\mathbf{NOT}}\rho)$ and $(\rho \bullet \sigma)$ is the abbreviation for $\mathbf{IAND}(\rho, \sigma)$.

Taking into account these conventions, we introduce the following algebraic system associated with the quantum gates:

Definition 3.6. The *Poincaré irreversible quantum computational algebra* (for short, Poincaré IQC-algebra) is given by

$$\langle \mathcal{D}(\mathbf{C}^2), \bullet, \oplus, \neg, \sqrt{\cdot}, P_0, P_1, \rho_{1/2} \rangle$$

P_0, P_1, and $\rho_{1/2}$ are distinguished elements of $\mathcal{D}(\mathbf{C}^2)$ representing privileged states associated with *true, false*, and *random* respectively. Further operation are defined as follows:

$$\alpha \odot \beta = \neg(\neg\alpha \oplus \neg\beta)$$

$$\alpha \to \beta = \neg\alpha \oplus \beta$$

$$\alpha \wedge \beta = \alpha \odot (\varphi \to \beta)$$

$$\alpha \vee \beta = (\alpha \to \beta) \to \beta$$

These operations satisfy the following properties:

**Lemma 3.7.** *Let $\tau, \upsilon \in \mathcal{D}(\mathbf{C}^2)$. Then we have:*

1. $\mathcal{D}(\mathbf{C}^2), \bullet\rangle$ *is an abelian monoid,*
2. $\tau \bullet P_0 = P_0$
3. $\tau \bullet P_1 = \rho_{p(\tau)}$
4. $p(\tau \bullet \upsilon) = p(\tau)p(\upsilon)$
5. $p(\tau \oplus \upsilon) = p(\tau) \oplus p(\upsilon)$
6. $\sqrt{\neg\tau} = \neg\sqrt{\tau}$
7. $\sqrt{\sqrt{\tau}} = \neg\tau$
8. $p(\sqrt{\tau \bullet \upsilon}) = p(\sqrt{\tau \oplus \upsilon}) = 1/2$
9. $\frac{p(\sigma)}{4} \oplus \frac{p(\sqrt{\sigma})}{4} \leq \frac{1+\sqrt{2}}{4\sqrt{2}}$, *if and only if $r_2^2 + r_3^2 \leq 1$ with $\sigma = \sigma(r_1, r_2, r_3)$*
10. $\frac{p(\sigma)}{4} \oplus \frac{1}{8} \leq \frac{3}{8} \leq \frac{1+\sqrt{2}}{4\sqrt{2}}$.

Proof: For 1–8, see Dalla Chiara *et al.* (2003, Lemma 6.2 and Lemma 5.3). Whereas we prove 9 as follows: We use the factor $1/4$ since for each $x, y \in [0, 1]$, $1/4x \oplus 1/4y = 1/4x + 1/4y$. By Proposition 3.2, $\frac{p(\sigma)}{4} + \frac{p(\sqrt{\sigma})}{4} = (1 - r_3)/8 + (1 - r_2)/8$. If we suppose that (r_2, r_3) lies in the unitary circle then this bound is given by simple problem of maximization with conditional extremes. To see the converse, taking into account Proposition 3.2,

$(1 - r_3)/2 + (1 - r_2)/2 \leq 1 + 1/\sqrt{2}$, thus $-(r_2 + r_3) \leq 2/\sqrt{2}$. Using polar co-ordinates $r_2 = r\cos\theta, r_3 = r\sin\theta$, thus we have $-r(\cos\theta + \sin\theta) \leq 2/\sqrt{2}$. But the maximum of $-r(\cos\theta + \sin\theta)$ is given when $\theta = 5/4\pi$, in this case $(\cos\theta + \sin\theta) = -2/\sqrt{2}$ and the $r \leq 1$. Thus, (r_2, r_3) lies in the unitary circle. Property 10 is immediate. $\square$

4. PRELIMINARIES ON UNIVERSAL ALGEBRA: MV-ALGEBRAS AND PMV-ALGEBRAS

In this section, we first deal with basic notions on universal algebra that we will apply to the PMV structures which are the algebraic counterpart of a type of enriched Łukaciewcz infinite valued calculus. This calculus will play a fundamental role in the logic associated with quantum gates. We recall from Balbes and Dwinger (1974) and Burris and Sankappanavar (1981) some basic tools of injectives and universal algebra, respectively. Let $\mathcal{A}$ be a class of algebras. For all algebras, A,B in $\mathcal{A}$, $[A, B]_{\mathcal{A}}$ will denote the set of all homomorphism $g : A \to B$. The class $\mathcal{A}$ is said to be a variety if and only if it is definable by equations or, equivalently, it is closed by subalgebras, direct product and homomorphic images. An algebra A in $\mathcal{A}$ is *injective* if and only if for every $f \in [B, A]_{\mathcal{A}}$ and every monomorphism $g \in [B, C]_{\mathcal{A}}$ there exists $h \in [C, A]_{\mathcal{A}}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
B & \xrightarrow{\ \ f\ \ } & A \\
{\scriptstyle g}\big\downarrow & {\equiv}\quad\nearrow & \\
 & {\scriptstyle h} & \\
C & &
\end{array}
$$
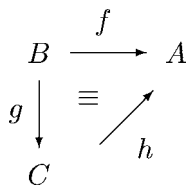
A is *self-injective* if and only if every homomorphism from a subalgebra of A into A, extends to an endomorphism of A. For each algebra A, we denote by $Con(A)$, the congruence lattice of A, the diagonal congruence is denoted by Δ and the largest congruence A^2 is denoted by ∇. An algebra I is *simple* if and only if $Con(I) = \{\Delta, \nabla\}$. An algebra A is *semisimple* if and only if it is a subdirect product of simple algebras. A nontrivial algebra T is said to be *minimal* in $\mathcal{A}$ if and only if for each nontrivial algebra A in $\mathcal{A}$, there exists a monomorphism $f : T \to A$. A simple algebra I_M is said to be *maximum simple* if and only if for each simple algebra I, I can be embedded in I_M. A simple algebra is *hereditarily simple* if and only if all its subalgebras are simple. An algebra A is *rigid* if and only if the identity homomorphism is the only automorphism. An algebra A has the *congruence extension property* (CEP) if and only if for each subalgebra B and $\theta \in Con(B)$ there is a $\phi \in Con(A)$ such that $\theta = \phi \cap A^2$. A variety $\mathcal{V}$ satisfies CEP if and only if every algebra in $\mathcal{V}$ has the CEP. It is clear that if $\mathcal{V}$ satisfies CEP then every simple algebra is hereditarily simple.

Definition 4.1. Let $\mathcal{V}$ be a variety. Two constant terms 0, 1 of the language of $\mathcal{V}$ are called *distinguished constants* if and only if $A \models 0 \neq 1$ for each nontrivial algebra A in $\mathcal{V}$.

Theorem 4.2. *Let $\mathcal{A}$ be a variety satisfying CEP, with distinguished constants 0, 1. If I is a self-injective maximum simple algebra in $\mathcal{A}$ then I is injective.*

Proof: (Freytes, 2004, Theorem 3.4). □

Definition 4.3. An MV-algebra (Cignoli *et al.*, 2000) is an algebra $\langle A, \oplus, \neg, 0 \rangle$ of type $\langle 2, 2, 0 \rangle$ satisfying the following equation:

MV1 $\langle A, \oplus, 0 \rangle$ is an abelian monoid,
MV2 $\neg\neg x = x$
MV3 $x \oplus \neg 0 = \neg 0$
MV4 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

We denote by $\mathcal{MV}$ the variety of MV-algebras. In agreement with the usual MV-algebraic operations we define

$$1 = \neg 0$$

$$x \odot y = \neg(\neg x \oplus \neg y)$$

$$x \rightarrow y = \neg x \oplus y$$

$$x \wedge y = x \odot (x \rightarrow y)$$

$$x \vee y = (x \rightarrow y) \rightarrow y$$

On each MV-algebra A we can define an order in A given by

$$x \leq y, \quad \text{if and only if } x \rightarrow y = 1$$

This order turns $\langle A, \wedge, \vee, 0, 1 \rangle$ into a distributive bounded lattice with 1 the greatest element and 0 the smallest element.

A very important example of MV-algebra is $[0, 1]_{MV} = \{[0, 1], \oplus, \neg, 0\}$ such that $[0, 1]$ is the real unit segment and $\oplus$ and $\neg$ are defined as follows:

$$x \oplus y = \min(1, x + y)$$

$$\neg x = 1 - x$$

The derivate operations in $[0, 1]_{MV}$ are given by $x \odot y = \max(0, x + y - 1)$ and $x \rightarrow y = \min(1, 1 - x + y)$. Finally the MV-lattice structure is the natural order in $[0, 1]$.

We recall from Hájek (1998) some well-known facts about implicative filters and congruences on MV-algebras. To do this, we will see MV-algebras as particular

case of BL-algebras. Let A be an MV-algebra and $F \subseteq A$. Then F is an *implicative filter* if and only if it satisfies the following conditions:

  1. $1 \in F$,
  2. if $x \in F$ and $x \rightarrow y \rightarrow \in F$ then $y \in F$.

It is easy to verify that a nonempty subset F of an MV-algebra A is an implicative filter if and only if for all $a, b \in A$:

  – If $a \in F$ and $a \leq b$ then $b \in F$,
  – if $a, b \in F$ then $a \odot b \in F$.

Note that an implicative filter F is proper if and only if 0 does not belong to F. The intersection of any family of implicative filters of A is again an implicative filter of A. We denote by $\langle X \rangle$ the implicative filter generated by $X \subseteq A$, i.e., the intersection of all implicative filters of A containing X. We abbreviate this as $\langle a \rangle$ when $X = \{a\}$ and it is easy to verify that $\langle X \rangle = \{x \in A : \exists w_1 \cdots w_n \in X$ such that $x \geq \odot \cdots \odot w_n\}$. For any implicative filter F of $A, \theta_F = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in F\}$ is a congruence on A. Moreover, $F = \{x \in A : (x, 1) \in \theta_F\}$. Conversely, if $\theta \in Con(A)$ then $F_\theta = \{x \in A : (x, 1) \in \theta\}$ is an implicative filter and $(x, y) \in \theta$ if and only if $(x \rightarrow x, 1) \in \theta$ and $(y \rightarrow x, 1) \in \theta$. Thus, the correspondence $F \rightarrow \theta_F$ is a bijection from the set of implicative filters of A onto the set $Con(A)$. If F is an implicative filter of A, we shall write A/F instead of A/θ_F, and for each $x \in A$ we shall write x/θ_F for the equivalence class of x. An implicative filter is called *prime* if and only if, for each $x, y \in A, x \rightarrow y \in F$ or $y \rightarrow x \in F$.

Proposition 4.4.   *Let A be an MV-algebra and F be an implicative filter in A. Then we have the following: A/F is tottaly ordered if and only if F is prime*

Proof: See Hájek (1998, Lemma 2.3.14).   □

Theorem 4.5.   *For every MV-algebra A the following conditions are equivalent:*

  1. *A is simple,*
  2. *A is no trivial and for every nonzero element $x \in A$, there is an integer $n > 0$ such that $1 = x \oplus \cdots \oplus x$ (n times),*
  3. *A is no trivial and for each element $x < 1$ in A, there is an integer $n > 0$ such that $0 = x \odot \cdots \odot x$ (n times),*
  4. *A is isomorphic a subalgebra of$[0, 1]_{MV}$.*

Proof: See Cignoli *et al.* (2000, Theorem 3.5.1).   □

Proposition 4.6. *Two subalgebras A, B of [0, 1]$_{MV}$ are isomorphic if and only if A = B; the identity function is the only automorphism of A*

Proof: See Cignoli *et al*. (2000, Corollary 7.2.6) □

Corollary 4.7.

1. *[0, 1]$_{MV}$ is the maximum simple algebra in $\mathcal{MV}$.*
2. *[0, 1]$_{MV}$ is hereditary simple algebra.*
3. *[0, 1]$_{MV}$ is self-injective.*

Definition 4.8. A product MV-algebra (Montagna, 2000) (for short, PMV-algebra) is an algebra $\langle A, \oplus, \bullet, \neg, 0 \rangle$ of type (2, 2, 1, 0) satisfying the following:

1. $(A, \oplus, \neg, 0)$ is an MV-algebra,
2. $(A, \bullet, 1)$ is an abelian monoid,
3. $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg(x \bullet z)$.

We denote by $\mathcal{PMV}$ the variety of PMV-algebras. An important example of PMV-algebra is $[0, 1]_{MV}$ equipped with multiplication. This algebra is denoted by $[0, 1]_{PMV}$. Note that every Boolean algebra becomes a PMV-algebra by letting the product operation coincide with the infimun operation. The following are almost immediate consequences of the definition.

Lemma 4.9. *In each PMV-algebra we have*

1. $0 \bullet x = 0$,
2. *If $a \leq b$ then $a \bullet x \leq b \bullet x$,*
3. $x \odot y \leq x \bullet y \leq x \wedge y$.

Definition 4.10. If we consider $1/2 \in [0, 1]$ then the algebra S is the subalgebra of $[0, 1]_{PMV}$ generated by 1/2

It is clear that, as a set, *S* is contained in the rationals of the real interval [0,1]. The algebra *S* and the following proposition will play an important role in the later logical treatment of irreversible quantum gates.

Proposition 4.11. $(S, \leq)$ *is dense order in the real interval [0, 1].*

Proof: Let $a, b \in [0, 1]$. We assume that $a < b < 1/2$. Let $n_0 \in N$ be the first natural such that $1/2^{n_0} \leq a$. Let $n_1 \in N$ such that $1/2^{n_1} \leq 1/4 \min\{b - a, a - 1/2^{n_0}\}$. Thus, there exists $n \in N$ such that

$$a < s = 1/2^{n_0} + \sum_n 1/2^{n_1} = 1/2^{n_0} + \bigoplus_n 1/2^{n_1} < b$$

and, by construction, $s \in S$. If $1/2a < b$, then $\neg b < \neg a < 1/2$ since $\neg 1/2 = 1 - 1/2$. Therefore, we find $s \in S$ such that $\neg b < s < \neg a$ following the same argument. Thus, $a < \neg s = 1 - s < b$. Finally, $(S, \leq)$ is a dense order in the real interval $[0, 1]$. □

Lemma 4.12. *Let A be a PMV-algebra and let B be the underlying MV-algebra. Then A and B have the same congruences.*

Proof: See Montagna (2000, Lemma 2.11). □

In view of this, implicative filters on a PMV-algebra A are identifiable to congruences of A, thus we use the notation A/F for the quotient PMV-algebra.

Corollary 4.13. *Let A be a PMV-algebra and F be an implicative filter. Then we have A/F as totally ordered if and only if F is a prime implicative filter.*

Proof: Follows from Lemma 4.4. □

Corollary 4.14. *If A is a subalgebra of $[0, 1]_{\text{PMV}}$, then A is simple.*

Proof: Follows from Theorem 4.5, Lemma 4.12. □

Theorem 4.15. *Let A be a semisimple MV-algebra. There is at most one operation $\bullet$ making $(M, \bullet)$ into a PMV-algebra.*

Proof: See Mundici and Riecǎn (2002, Theorem 3.1.14). □

Proposition 4.16. *If $\mathcal{A}$ is a subvariety of $\mathcal{PMV}$, then $\mathcal{A}$ satisfies CEP.*

Proof: Let A be a PMV-algebra and let B be a subalgebra of A. By Lemma 4.12, our proof is based only on the notion of implicative filters. For each implicative filter F of B, let $\langle F \rangle_A$ be the implicative filter of A generated by F. Clearly $F \subseteq \langle F \rangle_A$. To see the converse, let $b \in B \cap \langle F \rangle_A$. Then there exists $a_1, \ldots, a_n \in F$ such that $a_1 \odot a_2 \odot \ldots \odot a_n \leq b$. Since $b \in B$ and F is an implicative filter of B, hence upward closed, it follows that $b \in F$. □

Theorem 4.17. $[0, 1]_{\text{PMV}}$ *is injective in $\mathcal{PMV}$.*

Proof: It is clear that $0, 1$ are distinguished constant terms in $\mathcal{PMV}$. By Proposition 4.16, $\mathcal{PMV}$ satisfies CEP. We want to see that $[0, 1]_{\text{PMV}}$ is a maximum simple algebra in $\mathcal{PMV}$. Let A be a simple algebra in $\mathcal{PMV}$. By Lemma 4.12, A is simple as MV-algebra. Thus, by Corollary 4.7.1 there exists a sub-MV-algebra

A_0 of $[0, 1]_{MV}$ isomorphic to A. Let $i : A \to A_0$ an MV-isomorphism, then i induces an operation $\bullet$ on A_0 making $(A_0, \bullet)$ into a PMV-algebra. More precisely: If $x, y \in A_0$ such that $x = i(a)$ and $y = i(b)$ then $x \bullet y = i(a \bullet b)$.) Since A_0 is a simple MV-algebra (Corollary 4.7.3), by Theorem 4.15, the only possible candidate to be the $\bullet$ such that $(A_0, \bullet)$ turns to be a PMV-algebra is the usual product inherited from $[0, 1]$, because it is an abelian monoid with 1 as neutral element and satisfies $x(y \odot \neg z) = (xy) \odot \neg(xz)$.) Thus, for each $a, b \in A, i(a \bullet b) = i(a)i(b)$ resulting $i : A \to A_0 \subseteq [0, 1]_{PMV}$ a PMV-monomorphism. This proves that $[0, 1]_{PMV}$ is a maximum simple algebra. Using the last argument and Corollary 4.7.3, $[0, 1]_{PMV}$ is self-injective. Finally, we are within the hypothesis of Theorem 4.2, resulting $[0, 1]_{PMV}$ injective in $\mathcal{PMV}$.   □

5. MOTIVATION OF THE LOGICAL CALCULUS ASSOCIATED WITH IQC-ALGEBRA

In Dalla Chiara *et al.* (2003), it is said that "the logic arising from quantum computation represents, in a sense, a new example of fuzzy logic" and also that its axiomatization is an open problem. In this section, we begin to rigorously delineate the relation between the logic associated to the Poincaré structure and fuzzy logic from a semantic approach.

This will be done by establishing a logical propositional system in which propositions are interpreted in $\mathcal{D}(C^2)$ in a way that the semantic consequence $\models$ comes from the relation $p(\sigma) \leq p(\rho)$.

Recalling that the assignation of probability is a function $p : \mathcal{D}(C^2) \to [0, 1]$, it is possible to establish the following equivalence relation in $\mathcal{D}(C^2)$ given by

$$\sigma \equiv \nu, \quad \text{if and only if} \quad p(\sigma) = p(\nu)$$

If we denote by $[\sigma]$ the equivalence class of $\sigma \in \mathcal{D}(C^2)$, in view of Proposition 2.14 and Lemma 3.7, it is clear that

$$[\sigma] = [(\sigma \bullet P_1)] = [\rho_{P(\sigma)}]$$

Thus, we can consider the identification

$$(\mathcal{D}(C^2)/ \equiv) = (\rho_\lambda)_{\lambda \in [0,1]}$$

And the following proposition is easily demonstrated:

Proposition 5.1. $\langle (\mathcal{D}(C^2)/ \equiv), \oplus, \bullet, \neg, [P_0], [P_1] \rangle$ *is a PMV-algebra isomorphic to* $[0, 1]_{PMV}$. *The isomorphism is given by* $\rho_\lambda \to \lambda$. *The operations* $\vee, \wedge$ *give the PMV lattice structure in* $(\mathcal{D}(C^2)/ \equiv$.

This proposition shows that a logic associated with quantum gates would admit an algebraic counterpart strongly associated with PMV-algebras. In other

words, it would be a calculus based on the Łukaciewicz infinite valued calculus, enriched with a connective associated to **IAND**. This points a connection with fuzzy logic.

On the other hand, if we consider the finitely generated reduct of $\mathcal{D}(\mathbf{C}^2)$ given by $\langle P_0, P_1, \rho_{1/2}\bullet, \oplus, \neg\rangle$ we see that this reduct is a PMV-algebra isomorphic to the algebra S. In view of this, we can consider from a semantic level the Poincaré structure as equivalent to

$$\langle \mathcal{D}(\mathbf{C}^2), \bullet, \oplus, \neg, \sqrt{} , S\rangle$$

taking the elements of S as constant operations. We adopt this last structure as a natural model of the propositional calculus associated with quantum gates. This propositional calculus will relate the properties of the probability assignation with respect to the operations associated to the gates that model the connectives. We will show that both the notion of theorem and of syntactic consequence in this logic system are equivalent to theorems of the theory of PMV propositional calculus, thus establishing the rigorous connection with fuzzy logic.

We will also consider the following problem: given a theory T and a formula α, we want to give a kind of parameter that shows the *relevance degree* of T with respect to α. It may be interpreted as a dependence factor of a possible output with respect to an input that is known with certainty. This reflects precisely what a quantum computer does.

For this development we adopt a Pavelka style calculus (Hájek, 1998) in which the consideration of the elements of the subalgebra S as constants will show its importance since they will be useful to "measure" the mentioned dependence factor. In other sense, this shows a strong relation with approximate reasoning, which is another item considered within fuzzy logic. Under this perspective, we will establish a completeness theorem.

We will call *Irreversible Quantum Computational Logic* the logic associated with the irreversible Poincaré structure, for short *IQCL*.

6. THE *IQCL*-PROPOSITIONAL CALCULUS

We introduce here the basic tools and properties associated with *IQCL*.

Definition 6.1. The prepositional IQCL-language, i.e. the notion of *formulas*, is defined in the usual way from the following alphabet:

$$\langle P, \oplus, \odot, \rightarrow, \bullet, \vee, \wedge, \neg, \sqrt{}, \overline{S}, (,) \rangle$$

where $P = \{p_1, p_2, \dots, \}$ is the set of propositional variables; $\oplus, \odot, \rightarrow, \bullet, \vee, \wedge$ are binary connectives; $\neg, \sqrt{}$ are unary connectives, the set $\overline{S} = (\overline{s})_{s \in S}$ is the set of constant connectives representing the elements of the algebra S and (,) conform the punctuation symbols. We will refer to the elements of $P \cup \overline{S}$ as *atomic formulas*. In

particular, we denote by $\perp, 1/2, \top \in \overline{S}$ the syntactic representatives of $P_0, \rho_{1/2}, P_1$ of the algebra S. In addition, we introduce by definition the binary connective $\equiv$ as follows:

$$\alpha \equiv \beta, \quad \text{for } (\alpha \rightarrow \beta) \odot (\beta \rightarrow \alpha)$$

We can establish the notion of complexity of formulas as a function $Comp : IQCL \rightarrow N$ such that $Comp(\alpha) = 0$ if α is an atomic formula; $Comp(\alpha * \beta) = 1 + Comp(\alpha) + Comp(\beta)$ for each binary connective $*$; $Comp(\neg\alpha) = Comp\sqrt{\alpha} = 1 + Comp(\alpha)$.

To begin considering the semantic aspect of this logic, we introduce the definition of *model*:

Definition 6.2. A *model* of *IQCL* is a function $e : \mathcal{L} \rightarrow \mathcal{D}(\mathcal{C}^2)$ such that satisfies the following conditions:

1. If $*$ is a binary connective, then $e(\alpha * \beta) = e(\alpha) * e(\beta)$,
2. $e(\sqrt{\alpha}) = \sqrt{e(\alpha)}$,
3. $e(\neg\alpha) = \neg e(\alpha)$,
4. $e(\overline{s}) = \rho_s$ for each $\overline{s} \in (\overline{S})$,
5. $e(\perp) = P_0$,
6. $e(\top) = P_1$.

It is clear that if two models e, e' coincide over atomic formulas, then $e = e'$.

Definition 6.3.

1. If $e : IQCL \rightarrow \mathcal{D}(\mathcal{C}^2)$ is a model and $\alpha \in IQCL$ then, the *probability value* of α in e is given by $e_p(\alpha) = p(e(\alpha))$.
2. β *is a consequence of* $\alpha(\alpha \models \beta)$ if and only if $e_p(\alpha) \leq e_p(\beta)$ for each model e.

Remark 6.4. It is not very hard to see that $e_p(\neg\alpha) = \neg e_p(\alpha)$ and if $*$ is a binary connective then $e_p(\alpha * \beta) = e_p(\alpha) * e_p(\beta)$. Because of the definition of *model* we can see that if $\overline{s} \in \overline{S}$ and e^i, e^j are two models then $e^i_p(\overline{s}) = e^j_p(\overline{s})$. In view of this, for each $\overline{s} \in \overline{S}$ we refer to $p(\overline{s})$ as the probability value of $\overline{s}$ since the probability value over $\overline{s}$ is independent of the model.

Definition 6.5. A formula α is a *tautology* if and only if for each model e, $e_p(\alpha) = 1$

Proposition 6.6. *Let α, β be formulas. Then we have*

$$\alpha \models \beta \text{ if and only if } \alpha \rightarrow \beta \text{ is a tautology.}$$

Thus, the idea of logical consequence is syntactically strongly related to the implicative connective.

Definition 6.7. If we consider $\mathcal{D} = \{\rho = (0, r_2, r_3) : \rho \in \mathcal{D}(\mathbf{C}^2)\}$, then $e : IQCL \to \mathcal{D}$ are called *reduced models*.

Proposition 6.8. *Let e be a model. Then there exists a reduced model e' such that $e_p = e'_p$.*

Proof: By Proposition 3.2 the probability values of formulas in e depend on the coordinates r_2, r_3. If α is an atomic formula with $e(\alpha) = (r_1, r_2, r_3)$ we define e' as $e'(\alpha) = (0, r_2, r_3)$. Finally, we extend the model e' by an inductive argument on the complexity of formulas, obtaining $e_p = e'_p$. □

Remark 6.9. From the last proposition and since our semantics depends only on the probability values of formulas, we can restrict the use of models to reduced models.

Proposition 6.10. *Let e, e' be two reduced models such that for each atomic formula $\alpha = e_p(\alpha) = e'_p(\alpha)$ and $e_p(\sqrt{\alpha}) = e'_p(\sqrt{\alpha})$. Then, we have $e = e'$.*

Proof: Let α be an atomic formula. Suppose that $e(\alpha) = (0, r_2, r_3)$, and $e'(\alpha) = (0, r'_2, r'_3)$. Since $e_p(\alpha) = e'_p(\alpha)$ and $ep(\sqrt{\alpha}) = e'_p(\sqrt{\alpha})$ then, by Proposition 3.2, we have that $r_3 = r'_3$ and $r_2 = r'_2$. Finally, by an inductive argument on the complexity of formulas, it results that $e = e'$ □

Definition 6.11. The following formulas are axioms of the *IQCL*:
   Łukasiewicz axioms:

W1 $\alpha \to (\beta \to \alpha)$,
W2 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,
W3 $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,
W4 $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$.

   Equivalence operations axioms:

E1 $\alpha \odot \beta \equiv \neg(\neg\alpha \oplus \neg\beta)$,
E2 $\alpha \to \beta \equiv \neg(\alpha \odot \neg\beta)$,
E3 $\neg\alpha \equiv \alpha \to \bot$,
E4 $\alpha \wedge \beta \equiv \alpha \odot (\alpha \to \beta)$,
E5 $\alpha \vee \beta \equiv (\alpha \to \beta) \to \beta$,
E6 $\neg\bot \equiv \top$.

Product axioms:

P1 $(\alpha \bullet \beta) \to (\beta \bullet \alpha)$,
P2 $(\top \bullet \alpha) \equiv \alpha$,
P3 $(\alpha \bullet \beta) \to \beta$,
P4 $(\alpha \bullet \beta) \bullet \gamma \equiv (\beta \bullet \gamma)$,
P5 $\alpha \bullet (\beta \odot \neg\gamma) \equiv (\alpha \bullet \beta) \odot \neg(\alpha \bullet \gamma)$.

S axioms: for each $\overline{r}, \overline{t} \in \overline{S}$

S1 $\overline{r} \odot \overline{t} \equiv \overline{r \odot t}$,
S2 $\overline{r} \to \overline{t} \equiv \overline{r \to t}$,
S3 $\overline{r}, \bullet\overline{t} \equiv \overline{r \bullet t}$.

$\sqrt{NOT}$ axioms:

Q1 $\sqrt{\sqrt{\alpha}} \equiv \neg\alpha$,
Q2 $\sqrt{\neg\alpha} \equiv \neg\sqrt{\alpha}$,
Q3 If $*$ is a binary connective $\sqrt{\alpha * \beta} \equiv \overline{1/2}$,
Q4 $\sqrt{\overline{s}} \equiv 1/2$ for each $\overline{s} \in \overline{S}$,
Q5 $\{((\frac{\overline{1}}{4} \bullet \alpha) \oplus ((\frac{\overline{1}}{4} \bullet \sqrt{\alpha})) \to \overline{s} : s \geq (1 + \sqrt{2})/4\sqrt{2}\}$.

The unique deduction rule of *IQCL* is the *modus ponens* (MP).

We note that the axiom Q5 refers to the relation between density operators σ, $\sqrt{\sigma}$ with respect to the probability values they may take, i.e. between components r_2, r_3 (see Lemma 3.7). It is not very hard to see that

Lemma 6.12. *The axioms of IQCL are tautologies.*

A theory over *IQCL* is a set of formulas T. A *proof* in T is a sequence of formulas $\alpha_1 \cdots \alpha_n$ such that each member is either an axiom of *IQCL* or a member of T or follows from some preceding members of the sequence using *modus ponens*. $T \vdash \alpha$ means that α is *provable* in T, that is, α is the last formula of a proof from T. If $T = \emptyset$, we use the symbology $\vdash \alpha$ and, in this case, we will say that α is a *theorem of IQCL*. A model e of *IQCL* is a *model of a theory* T if and only if $e_p(\alpha) = 1$ for each $\alpha \in T$. In this case, we will use the notation $e_p(T) = 1$. We use $T \models \alpha$ or *semantic consequence* in the case in which when $e_p(T) = 1$ then $e_p(\alpha) = 1$ for each model e.

The following proposition is easily verified from the fact that the inference rule MP preserves the probability value 1.

Proposition 6.13. *Let* T *be a theory in IQCL and e be a model. If* $e_p(T) = 1$ *and* $T \vdash \varphi$ *then* $e_p(\varphi) = 1$

Remark 6.14. Note that axioms W1–W4 and E1–E6 conform the same propositional deductive system as the infinite valued Lukasiewicz calculus (Hájek, 1998, Chapter 3.1). We had to add the Equivalence axioms in order that our notion of model resulted well defined. In fact, if we had considered, for example, $\rightarrow$ and $\perp$ as primitive connectives and defined $\neg$ as $\alpha \rightarrow \perp$, then we would have had to require that $e_p(\sqrt{\neg\alpha}) = e_p(\sqrt{\alpha \rightarrow \perp}) = 1/2$. But this is not in general true except when $e(\alpha) = \rho\lambda$ for some λ.

Proposition 6.15. Let $\alpha, \beta, \gamma \in IQCL$. Then we have

Ł1 $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\beta \rightarrow \gamma))$,
Ł2 $(\alpha \odot \beta) \rightarrow \alpha$,
Ł3 $(\alpha \odot \beta) \rightarrow (\beta \rightarrow \alpha)$,
Ł4 $(\alpha \odot (\alpha \rightarrow \beta)) \rightarrow (\beta \odot (\beta \rightarrow \alpha))$,
Ł5 $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \odot \beta) \rightarrow \gamma)$,
Ł6 $((\alpha \odot \beta) \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))$,
Ł7 $((\alpha \rightarrow \beta) \rightarrow \gamma)) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma)$,
Ł8 $\perp \rightarrow \alpha$.

Proof: Follows from W1–W4, E1–E6 and Hájek (1998, Lemma 3.1.9, Corollary 3.1.16, and Definition 2.2.4) □

In view of Ł3 in the last proposition, we can introduce the following notation: for each formula α, is $\alpha \odot \cdots \odot \alpha$ (n-times).

Proposition 6.16. *Let $\alpha, \beta \in IQCL$ and T be a theory then we have*

1. $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$,
2. $\vdash \alpha \rightarrow (\beta \rightarrow (\alpha \odot \beta))$,
3. $\vdash (\alpha \rightarrow \beta) \rightarrow ((\alpha \odot \gamma) \rightarrow (\beta \odot \gamma))$,
4. $T \vdash \alpha \odot \beta$, *if and only if $T \vdash \alpha$ and $T \vdash \beta$,*
5. $T \vdash \alpha \equiv \beta$, *if and only if $T \vdash \alpha \rightarrow \beta$ and $T \vdash \beta \rightarrow \alpha$,*
6. $T \vdash \alpha \rightarrow \beta$ *and $T \vdash \beta \rightarrow \gamma$ then $T \vdash \alpha \rightarrow \gamma$,*
7. $\vdash T$,
8. $\vdash \alpha \rightarrow (T \odot \alpha)$,
9. $\vdash (T \rightarrow \alpha) \rightarrow \alpha$,
10. $\vdash \neg\neg\alpha \rightarrow \alpha$,
11. $\vdash (\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$,,
12. $\vdash (\alpha \rightarrow \beta) \rightarrow ((\alpha \oplus \gamma) \rightarrow (\beta \oplus \gamma))$,,
13. $\vdash ((\alpha \equiv \beta) \odot (\beta \equiv \gamma)) \rightarrow (\alpha \equiv \gamma)$,
14. $\vdash (\alpha \equiv \beta) \rightarrow ((\alpha \rightarrow \gamma) \equiv (\beta \rightarrow \gamma))$,
15. $\vdash (\alpha \equiv \beta) \rightarrow ((\gamma\neg\alpha) \equiv (\gamma \rightarrow \beta))$,

16. $\vdash (\alpha \to \beta)^n \lor (\beta \to \alpha)^n$ *for each* $n \in N$,

17. $\vdash (\alpha \to \beta) \to ((\gamma \bullet \alpha) \to (\alpha \bullet \beta))$,

18. $\vdash (\alpha \to \beta) \to ((\alpha \bullet \gamma) \to (\beta \bullet \gamma))$,

19. $\vdash (\alpha \odot \beta) \to (\alpha \bullet \beta)$.

Proof: For 1, see Hájek (1998, Lemma 2.2.7). For 2 and 3, see Hájek (1998, Lemma 2.2.8). Item 4 follows from 2 and Ł2. Item 5 follows from item 4. Item 6 follows from Ł1. For 7–9, see Hájek (1998, Lemma 2.2.14). For 10, see Hájek (1998, Lemma 3.1.6). For 11, see Hájek (1998, Lemma 2.2.12). Item 12 follows from 3, 11, and Axiom E1. For 13–15, see Hájek (1998, Lemma 2.2.16). For 16, see Hájek (1998, Lemma 2.2.24). Now we prove 17.

(1) $\vdash (\gamma \bullet (\alpha \odot \neg\beta)) \to ((\alpha \odot \neg\beta))$ By Axiom P3

(2) $\vdash ((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to (\gamma \bullet (\alpha \odot \neg\beta))$ By Axiom P5, Proposition 6.16-5

(3) $\vdash ((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to (\alpha \odot \neg\beta)$ By 1, 2, Proposition 6.16-6

(4) $\vdash ((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to (\alpha \odot \neg\beta)) \to (\neg(\alpha \odot \neg\beta) \to \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)))$ By Proposition 6.16-11

(5) $\vdash \neg(\alpha \odot \neg\beta) \to \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta))$ By MP 3 and 4

(6) $\vdash (\alpha \to \beta) \to \neg(\alpha \odot \neg\beta)$ By Axiom E2, Proposition 6.16-5

(7) $\vdash (\alpha \to \beta) \to \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta))$ By 5 and 6, Proposition 6.16-6

(8) $\vdash \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to ((\gamma \bullet \alpha) \to (\gamma \bullet \beta))$ By Axiom E2, Proposition 6.16-5

(9) $\vdash (\alpha \to \beta) \to ((\gamma \bullet \alpha) \to (\gamma \bullet \beta))$ By 7 and 8 and Proposition 6.16-6

     Item 18 follows from 13, 14, 15, and 17. Now we prove 19.

(1) $\vdash (\top \to \alpha) \to ((\top \bullet \beta) \to (\alpha \bullet \beta))$ By Proposition 6.16-17

(2) $\vdash ((\top \to \alpha) \to ((\top \bullet \beta) \to (\alpha \bullet \beta))) \to (((\top \to \alpha) \odot (\top \bullet \beta)) \to (\alpha \bullet \beta))$ By Ł5

(3) $\vdash ((\top \to \alpha) \odot (\top \bullet \beta)) \to (\alpha \bullet \beta)$ By MP 1 and 2

(4) $\vdash \alpha \to (\top \to \alpha)$ By Proposition 6.16-1

(5) $\vdash (\alpha \to (\top \to \alpha)) \to ((\alpha \odot (\top \bullet \beta)) \to ((\top \to \alpha) \odot (\alpha \bullet \beta)))$ By Proposition 6.16-3

(6) $\vdash (\alpha \odot (\top \bullet \beta)) \to ((\top \to \alpha) \odot (\top \bullet \beta)))$ By MP 4 and 5

(7) $\vdash (\alpha \odot (\top \bullet \beta)) \to (\alpha \bullet \beta)$ By 3, 6, Proposition 6.16-6

(8) $\vdash \beta \to (\top \bullet \beta)$ By Axiom P2, Proposition 6.16-5

(9) $\vdash (\beta \to (\top \bullet \beta)) \to ((\alpha \odot \beta) \to (\alpha \odot (\top \bullet \beta)))$ By Proposition 6.16-3

(10) $\vdash (\alpha \odot \beta) \to (\alpha \odot (\top \bullet \beta))$ By MP 8 and 9

(11) $\vdash (\alpha \odot \beta) \to (\alpha \bullet \beta)$ By 7, 10,   □

                     Proposition 6.16-6

Now we can establish a kind of deduction theorem for the *IQCL* calculus:

Theorem 6.17. *Let T be a theory and α, β be formulas. Then we have that if $T \cup \{\alpha\} \vdash \beta$ then there exists $n \in N$ such that $T \vdash \alpha^n \to \beta$*

Proof: See Hájek (1998, Lemma 2.2.18)   □

Definition 6.18.   A theory T is *inconsistent* if and only if $T \vdash \bot$; otherwise it is *consistent*.

Lemma 6.19.   *Let T be theory. Then we have*

     1. T is inconsistent if and only if $T \vdash \alpha$ for each formula α,
     2. $T \cup \{\alpha\}$ is inconsistent if and only if $T \vdash \neg\alpha^n$ for some $n \in N$
     3. If T is consistent then T does not prove $\bar{s}$ for each $\bar{s} \in \bar{S} - \{\top\}$.

Proof: For 1, see Hájek (1998, Lemma 2.2.21). For 2, see Hájek (1998, Lemma 2.2.22). Item 3 follows from the axiom S1, Proposition 4.5, and Proposition 4.12.   □

Definition 6.20.   Let T be a theory over *IQCL* and α be a formula

     1. The relevance degree of T over α is $\|\alpha\|_T = \wedge\{e_p(\alpha) : e_p(T) = 1\}$,
     2. The proof degree of α is $|\alpha|_T = \vee\{r \in S : T \vdash \bar{r} \to \alpha\}$.

Remark 6.21.   As we have already mentioned in the previous section, this relevance degree (also called *truth degree* in the frame of Pavelka style logics) represents the semantic relevance of a theory T with respect to a formula α, that is, the greatest lower bound of the probability values that α may take if all probability values of the formulas of T are known to be 1. Our purpose is to show the equality between relevance degree and proof degree, the *Pavelka style strong completeness theorem*. This equality will give a syntactic notion of the relevance degree in the sense that it is the lowest upper bound $r \in S$ such that $T \vdash \bar{r} \to \alpha$.

Lemma 6.22.   *Let T be a theory and α be a formula. Then we have the following:*

     1. If T admits a model, then $\|\bot\|_T = 0$. In the other case, $\|\bot\|_T = 1$,
     2. If T is inconsistent, then $\|\alpha\|_T = 1$ for each formula α,
     3. If $s \in S$ is such that $s < |\alpha|_T$ then $T \vdash \bar{s} \to \alpha$.

Proof: 1: If T admits a model the result is obvious. In the other case, $\{e_p(\perp) : e_p(T) = 1\} = \emptyset$ resulting $\|\perp\|_T = 1$. 2: If T is inconsistent, then $T \vdash \perp$, so T does not admit a model. Thus, for each formula α, $\{e_p(\alpha) : e_p(T) = 1\} = \emptyset$, resulting $\|\alpha\|_T = 1$. 3: If $s < |\alpha|_T$ then, by definition of $|\alpha|_T$, there exists $s_1 \in S$ such that $s < s_1 \leq |\alpha|_T$ and $T \vdash \bar{s}_1 \to \alpha$. By axioms S1 . . . S3 it is not very hard to see that $\vdash \bar{s} \to \bar{s}_1$. Finally, by Proposition 6.15-6, $T \vdash \bar{s} \to \alpha$ □

7. THE PMV-FRAGMENT OF IQCL

In order to establish the formal connection between *IQCL* and fuzzy logic via a Pavelka style strong completeness theorem, we will deal with a fragment of *IQCL* whose algebraic counterpart is a PMV-algebra so to be able to use the results obtained in Section 4.

Definition 7.1. We build the PMV-fragment, noted $IQCL_{\text{PMV}}$, from the sub-language of *IQCL* given by

$$\langle P \cup (\sqrt{p})_{p \in P}, \quad \oplus, \odot, \to, \bullet, \wedge, \vee, \neg, \overline{S}, (,) \rangle$$

We consider the subsystem of *IQCL* given by Axioms W1–W4, E1–E6, P1–P5, S1–S3 as the axiomatic system for IQCL$_{\text{PMV}}$.

In this section, we must take the formulas $(\sqrt{p})_{p \in P}$ as atomic formulas of the fragment. We denote by $\vdash_{\text{PMV}}$ deductions on the PMV-fragment. A valuation over IQCL$_{\text{PMV}}$ is a function $v : \text{IQCL}_{\text{PMV}} \to [0, 1]_{\text{PMV}}$ such that $u(\alpha * \beta) = v(\alpha) * v(\beta)$ for each binary connective $*$, $v(\neg\alpha) = \neg v(\alpha)$, and $v(\bar{s}) = s$ for each $\bar{s} \in \bar{S}$. Let T be a theory and α be a formula, both in IQCL$_{\text{PMV}}$. A formula α is called IQCL$_{\text{PMV}}$-*tautology* if and only if for each valuation v, $v(\alpha) = 1$. We define *relevance degree* in IQCL$_{\text{PMV}}$, $\|\alpha\|_T^{PMV}$ as in IQCL but in terms of the valuation over IQCL$_{\text{PMV}}$. In the same way, the *proof degree* $|\alpha|_T^{\text{PMV}}$ is defined in the PMV-fragment taking the axiomatic system of IQCL$_{\text{PMV}}$.

Remark 7.2. Note that Propositions 6.15 and 6.16 are also valid in the PMV-fragment.

Definition 7.3. Let T be a theory in $IQCL_{\text{PMV}}$. Then T is said to be *complete* if and only if, for each pair of formulas α, β, we have: $T \vdash_{\text{PMV}} \alpha \to \beta$ or $T \vdash_{\text{PMV}} \beta \to \alpha$.

Lemma 7.4. *Let T be a theory and α be a formula, both in IQCL$_{\text{PMV}}$. Suppose that T does not prove α in the PMV-fragment. Then there exists a consistent complete theory T' in IQCL$_{\text{PMV}}$ such that $T \subseteq T'$ and T' does not prove α in the PMV-fragment.*

Proof: See Hájek (1998, Lemma 2.4.2)   □

Theorem 7.5.   *Let T be theory over $IQCL_{PMV}$. For each formula α, we define $[\alpha] = \{\beta : T \vdash_{PMV} \alpha \equiv \beta\}$. Let $L_T = \{[\alpha] : \alpha \in IQCL_{PMV}\}$. If we define, for each binary connective $*$, the following operations in L_T:*

$$0 = [\bot],$$

$$1 = [\top],$$

$$\neg[\alpha] = [\neg\alpha],$$

$$[\alpha] * [\beta] = [\alpha * \beta].$$

then we have

   1. *$\langle L_T, \oplus, \odot, \to, \bullet, \wedge, \vee, \neg, 0, 1 \rangle$ is a PMV-algebra and $([\bar{s}])_{s \in S}$ is a sub-algebra isomorphic to the algebra S.*
   2. *If T is a complete theory then L_T is totally ordered.*

Proof: We first prove that the operations are well defined on L_T. In the case of $\oplus, \odot, \to, \wedge, \vee, \neg$, we refer to Hájek (1998, Lemma 2.3.12). Then we prove the definition of $\bullet$. Suppose that $[\alpha_1] = [\alpha_2]$ and $[\beta_1] = [\beta_2]$, we need to see that $[\alpha_1] \bullet [\beta_1] = [\alpha_2] \bullet [\beta_2]$ or equivalently $[\alpha_1 \bullet \beta_1] = [\alpha_2 \bullet \beta_2]$

(1) $T \vdash_{PMV} \alpha_1 \to \alpha_2$	By hypothesis, Proposition 6.16-5
(2) $T \vdash_{PMV} \beta_1 \to \beta_2$	By hypothesis, Proposition 6.16-5
(3) $\vdash_{PMV} (\alpha_1 \to \alpha_2) \to ((\alpha_1 \bullet \beta_1) \to (\alpha_2 \bullet \beta_2))$	By Proposition 6.16 16
(4) $T \vdash_{PMV} (\alpha_1 \bullet \beta_1) \to (\alpha_2 \bullet \beta_2)$	By MP 2 and 3
(5) $\vdash_{PMV} (\beta_1 \to \beta_2) \to ((\alpha_2 \bullet \beta_1) \to (\alpha_2 \bullet \beta_2))$	By Proposition 6.16-16
(6) $T \vdash_{PMV} ((\alpha_2 \bullet \beta_1) \to (\alpha_2 \bullet \beta_2)$	By MP 4 and 5
(7) $T \vdash_{PMV} ((\alpha_1 \bullet \beta_1) \to (\alpha_2 \bullet \beta_2)$	By Proposition 6.16-6

   In the same way, we can prove that $T \vdash_{PMV} (\alpha_2 \bullet \beta_2) \to (\alpha_1 \bullet \beta_1)$ resulting in $[\alpha_1 \bullet \beta_1] = [\alpha_2 \bullet \beta_2]$, and thus the well definition of $\bullet$. By Hájek (1998, Lemma 2.3.12), the reduct $\langle L_T, \oplus, \odot, \to, \wedge, \vee, \neg, 0, 1 \rangle$ is an MV-algebra. In view of this, if we consider the Axioms P1–P5, this algebra is also a PMV-algebra. By Axioms S1–S3 it results henceforth that $([\bar{s}])_{s \in S}$ is a subalgebra isomorphic to the algebra S. For the other items, we refer to Hájek (1998, Lemma 2.4.2) taking into account Lemma 4.13.   □

We will refer to algebra L_T as the *Lindenbaum algebra* associated to the theory T. Now we will establish a Pavelka style strong completeness theorem for the PMV-fragment:

Theorem 7.6. *Let T be a theory and α be a formula, both over IQCL$_{\mathrm{PMV}}$. Then we have*

$$|\alpha|_T = \|\alpha\|_T$$

Proof: If T is inconsistent, this result is trivial. Assume that T is consistent. We first prove that $|\varphi|_T$ is a lower bound of $\{v(\varphi) : v(T) = 1\}$. Let v be a valuation such that $v(T) = 1$. If $r \in S$ such that $T \vdash \bar{r} \to \varphi$, then we have that $1 = v(\bar{r} \to \varphi) = r \to v(\varphi)$. Thus, $r \le e(\varphi)$ and so $|\varphi|_T \le v(\varphi)$. We proceed now to prove that $|\varphi|_T$ is the greatest lower bound of $\{v(\varphi) : v(T) = 1\}$. In fact, let b be a lower bound of $\{v(\varphi) : v(T) = 1\}$. Suppose that $|\varphi|_T < b$. By Proposition 4.11, there exists $r_0 \in S$ such that $|\varphi|_T < r_0 < b$. Thus, T does not prove $\bar{r}_0 \to \varphi$. Then, by Proposition 7.4, there exists a complete consistent theory T' such that $T \subseteq T'$ and T' does not prove $\bar{r}_0 \to \varphi$. By Proposition 7.5, $L_{T'}$ is a totally ordered PMV-algebra containing $([\bar{s}])_{r \in S}$ as subalgebra isomorphic to S. Since T' does not prove $\bar{r}_0 \to \varphi$, then $[\bar{r}] \to [\varphi]$ in $L_{T'}$, and then $[\varphi] < [\bar{r}_0]$ since it is totally ordered. Let i_1 be the canonical embedding $([\bar{s}])_{r \in S} \to S \subseteq [0, 1]_{\mathrm{PMV}}$ and i_2 be the canonical embedding $([\bar{s}])_{r \in S} \to L'_T$. Since $[0, 1]_{\mathrm{PMV}}$ is an injective object in the variety $\mathcal{PMV}$, there exists an homomorphism $f : L_{T'} \to [0, 1]_{\mathrm{PMV}}$ such that the following diagram is commutative:

$$\bar{S} \xrightarrow{\ i_1\ } [0, 1]_{\mathrm{PMV}}$$
$$i_2 \downarrow \quad \equiv \nearrow f$$
$$L_{T'}$$

By commutativity of the diagram, $f([\varphi]) \le r_0 < b$. If we consider the natural projection $\pi : \mathrm{IQCL}_{\mathrm{PMV}} \to L_{T'}$ such that $\alpha \to [\alpha]$, then the composition $f\pi$ is a valuation over IQCL$_{\mathrm{PMV}}$ such that $f\pi(T') = 1$ and then $f\pi(T) = 1$ since $T \subseteq T'$ resulting $f\pi(\varphi) \in \{v(\varphi) : e(T) = 1\}$. But $f\pi(\varphi) \le r_0 < b$ which is a contradiction since b is a lower bound of $\{v(\varphi) : v(T) = 1\}$. Therefore $b \le |\varphi|_T$ resulting $|\varphi|_T = \|\varphi\|_T$. $\square$

8. COMPLETENESS OF *IQCL*

The study of the completeness of IQCL is performed from the strong completeness of the PMV-fragment using a translation of formulas from IQCL to IQCL$_{\mathrm{PMV}}$.

Definition 8.1. We define the PMV-*translation* as the function $\alpha \overset{t}{\to} \alpha_t$ such that:

1. $p \overset{t}{\to} p$ and $\sqrt{p} \overset{t}{\to} \sqrt{p}$ for each p atomic formula.
2. $\sqrt{\neg\alpha} \overset{t}{\to} (\neg\sqrt{\alpha})_t$.
3. $\sqrt{\sqrt{\alpha}} \overset{t}{\to} (\neg\alpha)_t$.
4. $\sqrt{\alpha * \beta} \overset{t}{\to} \overline{1/2}$ for each binary connective $*$.
5. $\alpha * \beta \overset{t}{\to} \alpha_t * \beta_t$ for each binary connective $*$.
6. $\neg\alpha \overset{t}{\to} \neg(\alpha_t)$.

If T is a theory in IQCL, we define the PMV-translation over the theory as the set $T_t = \{\alpha_t : \alpha \in T\}$.

By a simple induction on complexity of formulas, it is not very hard to see that the PMV-translation is a function $t : \text{IQCL} \to \text{IQCL}_{\text{PMV}}$. From the definition of the PMV-translation, we can immediately establish the following lemma:

Lemma 8.2. *Let α be a formula in IQCL. Then we have*

$$\vdash_{IQCL} \alpha \equiv \alpha_t$$

Taking into account the axiom Q5 of IQCL, we define the following theory in IQCL$_{\text{PMV}}$ which plays an important role in relation to deductions on IQCL with respect to deductions in IQCL$_{\text{PMV}}$:

Definition 8.3. We consider five groups of formulas in IQCL$_{\text{PMV}}$

$$T_1 = \left\{ \left(\left(\frac{\overline{1}}{4} \bullet p \right) \oplus \left(\left(\frac{\overline{1}}{4} \bullet \sqrt{p} \right) \right) \right) \to \overline{s} : p \in P, s \geq (1 + \sqrt{2})/4\sqrt{2} \right\}$$

$$T_2 = \left\{ \left(\left(\frac{\overline{1}}{4} \bullet \neg p \right) \oplus \left(\left(\frac{\overline{1}}{4} \bullet \neg\sqrt{p} \right) \right) \right) \to \overline{s} : p \in P, s \geq (1 + \sqrt{2})/4\sqrt{2} \right\}$$

$$T_4 = \left\{ \left(\left(\frac{\overline{1}}{4} \bullet \neg p \right) \oplus \left(\left(\frac{\overline{1}}{4} \bullet \neg\sqrt{p} \right) \right) \right) \to \overline{s} : p \in P, s \geq (1 + \sqrt{2})/4\sqrt{2} \right\}$$

$$T_3 = \left\{ \left(\left(\frac{\overline{1}}{4} \bullet p \right) \oplus \left(\left(\frac{\overline{1}}{4} \bullet \sqrt{p} \right) \right) \right) \to \overline{s} : p \in P, s \geq (1 + \sqrt{2})/4\sqrt{2} \right\}$$

$$T_5 = \left\{ \left(\left(\frac{\overline{1}}{4} \bullet \alpha \right) \oplus \frac{\overline{1}}{8} \right) \to \overline{s} : s \geq 3/8 = 1/4 \oplus 1/8 \right\}$$

Then we define $T_{Q5} = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$.

Lemma 8.4. *Let $\alpha \in IQCL$ and $s \in S$.*

If $s \geq (1 + \sqrt{2})/4\sqrt{2}\}$ then

1. $T_{Q5} \vdash_{\text{PMV}} \left(\left(\left(\frac{\bar{1}}{4} \bullet \alpha\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\alpha}\right)\right) \to \bar{s}\right)_t$ *noted* $T_{Q5} \vdash_{\text{PMV}} \alpha_t^1$

2. $T_{Q5} \vdash_{\text{PMV}} \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\alpha\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\sqrt{\alpha}\right)\right) \to \bar{s}\right)_t$ *noted* $T_{Q5} \vdash_{\text{PMV}} \alpha_t^2$

3. $T_{Q5} \vdash_{\text{PMV}} \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\alpha\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\alpha}\right)\right) \to \bar{s}\right)_t$ *noted* $T_{Q5} \vdash_{\text{PMV}} \alpha_t^3$

4. $T_{Q5} \vdash_{\text{PMV}} \left(\left(\left(\frac{\bar{1}}{4} \bullet \alpha\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\sqrt{\alpha}\right)\right) \to \bar{s}\right)_t$ *noted* $T_{Q5} \vdash_{\text{PMV}} \alpha_t^4$

If $s \geq 3/8 = \overline{1/4} \oplus 1/8$

5. $T_{Q5} \vdash_{\text{PMV}} \left(\left(\left(\frac{\bar{1}}{4} \bullet \alpha\right) \oplus \frac{\bar{1}}{8}\right) \to \bar{s}\right)_t$ *noted* $T_{Q5} \vdash_{\text{PMV}} \alpha_t^5$

Proof: We use induction on complexity of α. If $Comp(\alpha) = 0$, we consider two cases: *Case $\alpha \in \bar{S}$*: By axiom Q4, $\sqrt{\alpha} \equiv \overline{1/2}$ and by axiom S2, $\neg\overline{1/2} \equiv \overline{1/2}$. The result follows by this and using Proposition 6.16. *Case $\alpha \in P$*: the translation of five formulas is over T_{Q5}. Suppose that this result is valid when $Comp(\alpha) < n$. Let α be such that $Comp(\alpha) = n$

Suppose α is $\neg\beta$.

1. $\alpha_t^1 = \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\beta\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\neg\beta}\right)\right) \to \bar{s}\right)_t = \left(\left(\frac{\bar{1}}{4} \bullet \neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} = \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\beta\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\left(\sqrt{\beta}\right)\right)\right) \to \bar{s}\right)_t = \beta_t^2$. By inductive hypothesis $T_{Q5} \vdash_{\text{PMV}} \beta_t^2$.

2. $\alpha_t^2 = \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\neg\beta\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\sqrt{\neg\beta}\right)\right) \to \bar{s}\right)_t = \left(\left(\frac{\bar{1}}{4} \bullet \neg\neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s}$ But by using Proposition 6.16 $\vdash_{\text{PMV}} \left(\left(\frac{\bar{1}}{4} \bullet \neg\neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} \equiv \left(\left(\frac{\bar{1}}{4} \bullet \beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s}$ being $\left(\left(\frac{\bar{1}}{4} \bullet \beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} = \beta_t^1$. By inductive hypothesis we have $T_{Q5} \vdash_{\text{PMV}} \beta_t^1$.

3. $\alpha_t^3 = \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\neg\beta\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\neg\beta}\right)\right) \to \bar{s}\right)_t = \left(\left(\frac{\bar{1}}{4} \bullet \neg\neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s}$ But by using Proposition 6.16 $\vdash_{\text{PMV}} \left(\left(\frac{\bar{1}}{4} \bullet \neg\neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} \equiv \left(\left(\frac{\bar{1}}{4} \bullet \beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s}$ being $\left(\left(\frac{\bar{1}}{4} \bullet \beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} = \beta_t^4$. By inductive hypothesis we have $T_{Q5} \vdash_{\text{PMV}} \beta_t^4$.

4. $\alpha_t^4 = \left(\left(\left(\frac{\bar{1}}{4} \bullet \neg\beta\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\sqrt{\neg\beta}\right)\right) \to \bar{s}\right)_t = \left(\left(\frac{\bar{1}}{4} \bullet \neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s}$ But by using Proposition 6.16 $\vdash_{\text{PMV}} \left(\left(\frac{\bar{1}}{4} \bullet \neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg\neg\left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} \equiv \left(\left(\frac{\bar{1}}{4} \bullet \neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s}$ being $\left(\left(\frac{\bar{1}}{4} \bullet \neg\beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \left(\sqrt{\beta}\right)_t\right)\right) \to \bar{s} = \beta_t^3$. By inductive hypothesis we have $T_{Q5} \vdash_{\text{PMV}} \beta_t^3$.

5. Immediate, since it results an element of T_5

Suppose α is $\sqrt{\beta}$.

1. $\alpha_t^1 = \left(\left(\left(\frac{\bar{1}}{4} \bullet \sqrt{\beta}\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\sqrt{\beta}}\right)\right) \to \bar{s}\right)_t = \left(\left(\frac{\bar{1}}{4} \bullet \sqrt{\beta_t}\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg \sqrt{\beta_t}\right)\right) \to \bar{s}$
   But by using Proposition 6.16 $\vdash_{\text{PMV}} \left(\left(\frac{\bar{1}}{4} \bullet \sqrt{\beta_t}\right) \oplus \left(\frac{\bar{1}}{4} \bullet \neg \sqrt{\beta_t}\right)\right) \to \bar{s} \equiv$
   $\left(\left(\frac{\bar{1}}{4} \bullet \neg \beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\beta_t}\right)\right) \to \bar{s}$ being $\left(\left(\frac{\bar{1}}{4} \bullet \neg \beta_t\right) \oplus \left(\frac{\bar{1}}{4} \bullet \sqrt{\beta_t}\right)\right) \to \bar{s} =$
   β_t^3. By inductive hypothesis we have $T_{Q5} \vdash_{\text{PMV}} \beta_t^3$.

   For the rest of this case, it follows in a similar manner.

Suppose that α is $\beta * \gamma$ for $*$ a binary connective. Then we use the same procedure as for the atomic case for elements of $\overline{S}$. □

The following theorem establishes the relation between the deductive system of IQCL and the deductive system IQCL$_{\text{PMV}}$:

Theorem 8.5. *Let T be a theory and α be a formula both in IQCL. Then we have*

$$\top \vdash_{\text{IQCL}} \alpha \text{ if and only if } T_t \cup T_{Q5} \vdash_{\text{PMV}} \alpha_t$$

Proof: Suppose that $T \vdash_{\text{IQCL}} \alpha$. We use induction on the length of the proof of α noted by $Length(\alpha)$. If $Length(\alpha) = 1$, then we have the following possibilities:

1. α is one of axioms W1–W4, E1–E6, P1–P5, S1–S3. In this case, α_t results an axiom of the IQCL$_{\text{PMV}}$.
2. α is one of the axioms Q1–Q4. In this case, α_t looks like $\beta \equiv \beta$ in IQCL$_{\text{PMV}}$ and, by Propositions 6.16-1 and -9, these formulae are PMV-theorems.
3. If α is axiom Q5, then we use Lemma 8.4 resulting in $T_{Q5} \vdash \alpha_t$.
4. If $\alpha \in T$, it is clear that $\alpha_t \in T_t$.

Suppose that the theorem is valid for $Length(\alpha) < n$. We consider $Length(\alpha) = n$. Thus, we have an IQCL-proof α from T as follows:

$$\alpha_1, \ldots, \alpha_m \to \alpha, \ldots, \alpha_m, \ldots, \alpha_{n-1}, \alpha$$

obtaining α by MP from $\alpha_m \to \alpha$ and α_m. Using inductive hypothesis we have $T_t \cup T_{Q5} \vdash_{\text{PMV}} (\alpha_m \to \alpha)_t$ and $T_t \cup T_{Q5} \vdash_{\text{PMV}} (\alpha_m)_t$. Taking into account that $(\alpha_m \to \alpha)_t = (\alpha_m)_t \to \alpha_t$, by MP we have $T_t \cup T_{Q5} \vdash_{\text{PMV}} \alpha_t$. The converse is immediate from Lemma 8.2 and the fact that the formulas in T_{Q5} are IQCL-theorems.

Corollary 8.6. *Let $\alpha \in IQCL$, then we have*

$$\vdash_{IQCL} \alpha \text{ if and only if } T_{Q5} \vdash_{\text{PMV}} \alpha_t$$

Remark 8.7. Theorem 8.5 shows in a formal sense the syntactic relation between IQCL and fuzzy logic. More precisely, we recall that provable formulas from IQCL-theories are identifiable to provable formulas from PMV-theories obtained by translation plus T_{Q5}. Note that IQCL-theorems are PMV-theorems of T_{Q5}. Another important result that arises from these theorems is that the $\sqrt{}$ connective has "importance" only when applied to atomic formulas. More precisely, the real difficulty of $\sqrt{}$ in IQCL is captured by the PMV-theory T_{Q5}.

Now we introduce the following sets in order to study the relation between reduced models of IQCL and valuations of the PMV-fragment:

$$E_R = \{e : e \text{ is a reduced model of IQCL}\}$$

$$V_{Q5} = \{v : v \text{ is a valuation of IQCL}_{\text{PMV}} \text{ with } v(T_{Q5}) = 1\}$$

Proposition 8.8. *There exists a bijection $E_R \to V_{Q5}$, $e \to v_e$, such that $e_p(\alpha) = v_e(\alpha_t)$.*

Proof: Let $e \in E_R$ and let $v_e = e_p|\text{IQCL}_{\text{PMV}}$. We will see that $e \to v_e$ is well defined in the sense that $v_e \in V_{Q5}$. Let $\alpha \in T_{Q5}$. If $\alpha \notin T_5$, then it is not very hard to see that the density operator $e(\alpha)$ has the form $e(\alpha) = (\frac{\sigma}{4} \oplus \frac{\sqrt{\sigma}}{4}) \to s$ for some density operator σ and $s \geq \frac{1+\sqrt{2}}{4\sqrt{2}}$. But by Lemma 3.7, $p(\frac{\sigma}{4} \oplus \frac{\sqrt{\sigma}}{4}) \leq \frac{1+\sqrt{2}}{4\sqrt{2}}$. Thus, $e_p(\alpha) = 1$. If $\alpha \in T_5$ we use the same argument with the bound 3/8. Thus, the assignation is well defined. The injectivity follows for Proposition 6.10. Now we will prove the subjectivity. Let $v \in V_{Q5}$. For each atomic formula α let $e(\alpha) = ((1 - 2v(\alpha)), (1 - 2v(\sqrt{\alpha})))$. Thus, e is well defined over atomic formulas since v satisfies T_{Q5}. Then we can extend e to IQCL (with unicity, from Proposition 6.10) and it is clear that $e_p = v$ since e_p and v coincide over atomic formulas of IQCL$_{\text{PMV}}$. Finally, for each IQCL formula α, $e_p(\alpha) = v_e(\alpha_t)$ taking into account the inductive argument on the complexity of formulas and translation. $\square$

Remark 8.9. This last proposition shows the rigorous semantic connection between IQCL and fuzzy logic. More precisely, models of IQCL are identifiable to PMV-valuations that satisfy T_{Q5}.

Proposition 8.10. *Let T be a theory and α be a formula, both in IQCL. Then we have*

1. $|\alpha|_T^{\text{IQCL}} = |\alpha_t|_{T_t \cup T_{Q5}}^{\text{PMV}}$
2. $\|\alpha\|_T^{\text{IQCL}} = |\alpha_t|_{T_t \cup T_{Q5}}^{\text{PMV}}$

Proof: Item 1 follows from Theorem 8.5. Item 2 follows from Proposition 8.8. □

Now we can establish a Pavelka style strong completeness theorem for IQCL.

Theorem 8.11. *Let T be a theory and α be a formula both in IQCL. Then we have*

$$|\alpha|_T^{\text{IQCL}} = \|\alpha\|_T^{\text{IQCL}}$$

Proof: Follows from Proposition 8.10 and Theorem 7.6. □

Corollary 8.12. *Let T be a theory and α be a formula both in IQCL. If $\|\alpha\|_T^{\text{IQCL}} = 0$, then for each $s \in S, \overline{s} \to \alpha$ in not provable from T.*

From the above completeness theorem, we can establish a somehow compactness theorem.

Theorem 8.13. *Let T be a theory and α be a formula both in IQCL. Then we have*

$$If\ r \leq \|\alpha\|_T^{\text{IQCL}}\ then\ \exists\ T_0 \subseteq T\ finite\ such\ that\ r\ \leq \|\alpha\|_{T_0}^{\text{IQCL}}$$

Proof: If $r \leq \|\alpha\|_T^{\text{IQCL}}$ then by Theorem 8.11 and Lemma 6.22 we have $T \vdash_{\text{IQCL}} \overline{r} \to \alpha$. If $\alpha_1 \cdots \alpha_n, \overline{r} \to \alpha$ is a proof of $\overline{r} \to \alpha$ from T, we can consider the finite set T_0 given as $T_0 = \{\alpha_\kappa \in T : \alpha_\kappa \in \{\alpha_1, \ldots, \alpha_n\}\}$. Using again Theorem 8.11 we have $r \leq |\alpha|_{T_0}^{\text{IQCL}} = \|\alpha\|_{T_0}^{\text{IQCL}}$ □

9. CONCLUSIONS

The logical support associated with quantum treatment of information based on the Poincaré structure turns to be a particular case of the propositional theories related to fuzzy logic, more precisely to a type of infinite valued logic. This logical frame allows to use approximate reasoning of fuzzy logic as a tool for the study of the relevance that input data that are known with certainty have on the possible outputs.

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